# Hopf algebroids (and Lie bialgebroids) as gauge symmetries in 3D gravity

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Work in progress . . .

Hopf algebroids and NCG workshop

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# I. Classical and Quantum symmetries

# Gauge symmetries

## Correspondence: Gauge symmetries ↔ Algebra

| Type | Algebraic structure |
|---|---|
| Classical | (Poisson)-Lie groups |
| Semi-classical | (Classical r-matrices) Lie bialgebras |
| Quantum | (Quantum Groups) Hopf algebras (Quantum R-matrix) Today |

# Generalized gauge symmetries

## Correspondence: Generalized gauge symmetries ↔ Algebra

| Type | Algebraic structure |
|---|---|
| Classical | (?) Lie Groupoids |
| Semi-classical (As in prof. Xu talk) | (Classical dynamical r-matrices) Lie bialgebroids |
| Quantum | Today; Hopf algebroids (Quantum dynamical r-matrices) |

# II. Symmetries in 3D Classical and Quantum gravity

# 3d gravity as a Chern-Simons theory

## Local model spacetimes and isometry groups ($G_{\Lambda,c}$)

| $\Lambda$ | Euclidean ($c^2 < 0$) | Lorentzian ($c^2 > 0$) |
|---|---|---|
| $0$ | $\mathbf{E}^3 = \mathrm{ISO}(3)/\mathrm{SO}(3)$ | $\mathbf{M}^{2+1} = \mathrm{ISO}(2,1)/\mathrm{SO}(2,1)$ |
| $> 0$ | $\mathbf{S}^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$ | $\mathbf{dS}^{2+1} = \mathrm{SO}(3,1)/\mathrm{SO}(2,1)$ |
| $< 0$ | $\mathbf{H}^3 = \mathrm{SO}(3,1)/\mathrm{SO}(3)$ | $\mathbf{AdS}^{2+1} = \mathrm{SO}(2,2)/\mathrm{SO}(2,1)$ |

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## Just 6 possible Lie algebras $\mathfrak{g}_{\Lambda,c}$ generated by $\{J_0, J_1, J_2, P_0, P_1, P_2\}$

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad \text{and} \quad [P_a, P_b] = (\underbrace{-c^2\Lambda}_{\lambda}) \epsilon_{abc} J^c$$

# 3d gravity as a Chern-Simons theory (Witten 1988)

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## *Ad*-invariant (standard) symmetric bilinear form

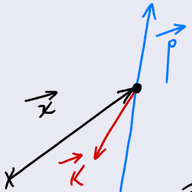
$$\langle J_a, J_b \rangle = 0, \quad \langle J_a, P_b \rangle = c^2 \eta_{ab} \quad \text{and} \quad \langle P_a, P_b \rangle = 0$$

# Why Quantum groups in 3D quantum gravity?

$G = ISO(2,1) = SL(2,\mathbb{R}) \ltimes \mathbb{R}^3$

## Phase space of a free point particle

(free point → relativistic)




$\vec{p}\cdot\vec{p} = m^2 \qquad \vec{K}\cdot\vec{p} = ms \qquad \vec{K} = \vec{x}\wedge\vec{p} + s\hat{p}$

(Lie) Geometric approach

$ISO(2,1) \ni (v,x)(mJ_0 + sP_0)(v,x)^{-1} = p_a J^a + K_a P^a$

Action (de Soussa 82)

$I = \int d\tau\, p_a \dot{x}^a + s\langle P_0, v^{-1}\dot{v}\rangle = \int d\tau \langle mJ_0 + sP_0, g^{-1}\dot{g}\rangle$

Poisson structure (KKS)

$\{K_a, K_b\} = -\epsilon_{abc}K^c, \quad \{K_a, p_b\} = -\epsilon_{abc}p^c, \quad \{p_a, p_b\} = 0$

# Why Quantum groups in 3D quantum gravity?

## Phase space of a gravitational point particle

Symplectic Leaves of $P_3^* \; (= SL(2,\mathbb{R}) \ltimes \mathbb{R}^3)$

$|||$

Conjugacy classes in $P_3$

$$(v,x)\, e^{-mJ_0 - sP_0} (v,x)^{-1} = (u, -8\pi G Ad(u^{-1}) j) \to (u, -8\pi G j)$$

$$u = v^{-1} e^{-mJ_0} v = e^{-8\pi G p_a J^a}$$

$$K = j = [x, p_a J^a] + s \hat{p}_a P_a + O(p^2)$$

Poisson structure

$$\{j_a, j_b\} = -\epsilon_{abc} j^c, \quad \{j_a, p_b\} = -\epsilon_{abc} p^c, \quad \{p_a, p_b\} = 0$$

# Why Quantum groups in 3D quantum gravity?

## Quantization of the dynamics (Quantum double and $\kappa$-Poincaré...)

$$[J_a, J_b] = \hbar \varepsilon_{abc} J^c, \quad [J_a, P_b] = \hbar \varepsilon_{abc} P^c$$

$$[P_a, P_b] = 0$$

Free Particle — Symmetry: $U(sl(2,\mathbb{R})) \ltimes Pol((\mathbb{R}^*)^3)$

↓

Gravitation Particle — Symmetry: $U(sl(2,\mathbb{R})) \ltimes Pol(SL(2,\mathbb{R}))$

# Why Quantum groups in 3D quantum gravity?

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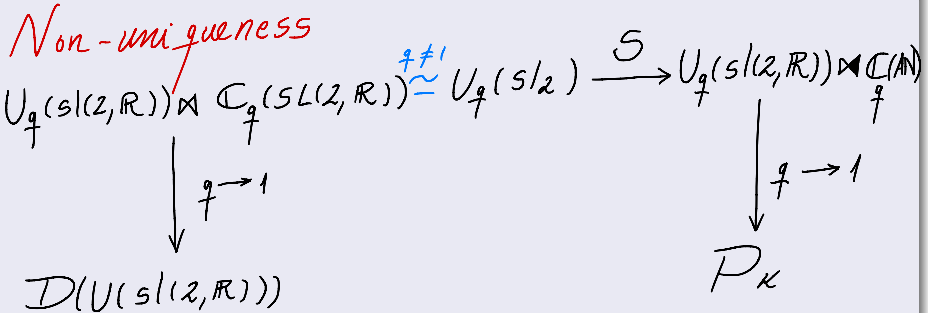


## Conclusion 1

**Quantum groups could be used to encode symmetries of 3D quantum gravity**

# III. Enlarging the structure of quantum symmetries: Quantum double

# Generalized FRST construction

## Input

- $\mathfrak{g}$ a finite dimensional Lie algebra and $\mathfrak{h}$ a Lie subalgebra.
- $V$ a finite dimensional vector space with basis $\{v_x\}_{x\in X}$.
- $\omega : X \to \mathfrak{h}^*$ an arbitrary map.
- $R \in \mathrm{M}_{\mathfrak{h}^*} \otimes \mathrm{End}_{\mathfrak{h}}(V \otimes V)$ a solution of the QDYBE, s.t. $R^{ab}_{xy} = 0$ if $\omega(x) + \omega(y) \neq \omega(a) + \omega(b)$

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## Output (FRST construction) [Koeling and Van Norden (2001)]

An $\mathfrak{h}$-bialgebroid generated by $\{L_{xy}\}_{x,y \in X}$ and two copies of $M_{\mathfrak{h}^*}$

# The Hopf algebroid $\mathfrak{F}_q(\mathfrak{sl}_2)$

**Definition (The Hopf algebroid $\mathfrak{F}_q(\mathfrak{sl}_2)$)**

Applying the construction above for $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{h} = \mathbb{C}$, $X = \{\pm\}$, $\omega(\pm) = \pm 1$ and

$$R_q(\lambda) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \frac{q^{-1}-q}{q^{2(\lambda+1)}-1} & 0 \\ 0 & \frac{q^{-1}-q}{q^{-2(\lambda+1)}-1} & \frac{(q^{2(\lambda+1)}-q^2)(q^{2(\lambda+1)}-q^{-2})}{(q^{2(\lambda+1)}-1)^2} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

Koeling and Van Norden constructed a Hopf algebroid (quantum dynamical group) denoted by $\mathfrak{F}_q(\mathfrak{sl}_2)$.

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## Notation

$$\alpha = L_{++}, \quad \beta = L_{+-}, \quad \gamma = L_{-+}, \quad \delta = L_{--}$$

# The Hopf algebroid $\mathfrak{F}_q(\mathfrak{sl}_2)$ (Product)

## Multiplication

$$
\begin{cases}
\alpha\beta = qF(\rho-1)\beta\alpha, & \alpha\gamma = qF(\lambda)\gamma\alpha \\
\beta\delta = qF(\lambda)\delta\beta, & \gamma\delta = qF(\rho-1)\delta\gamma \\
\alpha\delta - \delta\alpha = H(\lambda,\rho)\gamma\beta, & \beta\gamma - G(\lambda)\gamma\beta = I(\lambda,\rho)\alpha\delta
\end{cases}
$$

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$$\beta\delta = qF(\lambda)\delta\beta, \qquad \gamma\delta = qF(\rho-1)\delta\gamma$$
$$\alpha\delta - \delta\alpha = H(\lambda,\rho)\gamma\beta, \qquad \beta\gamma - G(\lambda)\gamma\beta = I(\lambda,\rho)\alpha\delta$$

## Functions

$$F(\lambda) = \frac{q^{2(\lambda+1)} - q^{-2}}{q^{2(\lambda+1)} - 1}, \qquad G(\lambda) = \frac{(q^{2(\lambda+1)} - q^{2})(q^{2(\lambda+1)} - q^{-2})}{(q^{2(\lambda+1)} - 1)^2}$$
$$H(\lambda,\rho) = \frac{(q-q^{-1})(q^{2(\lambda+\rho+2)} - 1)}{(q^{2(\lambda+1)} - 1)(q^{2(\rho+1)} - 1)},$$
$$I(\lambda,\rho) = \frac{(q-q^{-1})(q^{2(\rho+1)} - q^{2(\lambda+1)})}{(q^{2(\lambda+1)} - 1)(q^{2(\rho+1)} - 1)}$$

Limits $\lambda, \rho \to -\infty$ (recover Known HA)

# The Hopf algebroid $\mathfrak{F}_q(\mathfrak{sl}_2)$ (Determinant condition)

**Important comment!** ✓

In order to get a Hopf-algebroid it is required to adjoin the relation

$$\boxed{\alpha\delta - qF(\lambda)\gamma\beta = 1}$$

Hopf-algebroid analog of $\mathbb{C}_q[SL_2]$ + determinant condition ← Recovered for $\lambda \to -\infty$

# The Hopf algebroid $\mathfrak{F}_q(\mathfrak{sl}_2)$ (Counit, Coproduct and Antipode)

## Counit

$$\epsilon(\alpha) = T_{-1}, \quad \epsilon(\beta) = 0, \quad \epsilon(\gamma) = 0, \quad \epsilon(\delta) = T_{+1}, \quad \epsilon(f(\lambda \text{ or } \rho)) = f$$

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## Coproduct

$$\begin{aligned} \triangle(\alpha) &= \alpha \otimes \alpha + \beta \otimes \gamma, & \triangle(\beta) &= \alpha \otimes \beta + \beta \otimes \delta, \\ \triangle(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma, & \triangle(\delta) &= \gamma \otimes \beta + \delta \otimes \delta, \\ \triangle(f(\lambda)) &= f(\lambda) \otimes 1, & \triangle(f(\rho)) &= 1 \otimes f(\rho) \end{aligned}$$

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## Antipode

$$S(\alpha) = \frac{F(\lambda)}{F(\rho)}\delta, \quad S(\beta) = -\frac{q^{-1}}{F(\mu)}\beta, \quad S(\gamma) = -qF(\lambda)\gamma, \quad S(\delta) = \alpha$$

# $\mathfrak{F}_q(\mathfrak{sl}_2)$ as a deformation of $U_q(\mathfrak{sl}_2)$

Defined as the free algebra over the ring $\mathbb{C}[[\hbar]]$ with generators $H$ and $X_\pm$, such that

## Product

$$[H, X_\pm] = \pm 2X_\pm, \qquad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad \text{where } q \equiv e^{\frac{\hbar}{2}}.$$

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Standard notation

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## Coproduct

$$\triangle(H) = H \otimes 1 + 1 \otimes H, \qquad \triangle(X_\pm) = q^{-\frac{H}{2}} \otimes X_\pm + X_\pm \otimes q^{\frac{H}{2}}$$

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$$\epsilon(H) = \epsilon(X_\pm) = 0$$

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$$\epsilon(H) = \epsilon(X_\pm) = 0$$

## Antipode

$$S(H) = -H, \qquad S(X_\pm) = -q^{\pm 1} X_\pm$$

# $\mathfrak{F}_q(\mathfrak{sl}_2)$ as a deformation of $U_q(\mathfrak{sl}_2)$ [Due to Rosengren]

**Proposition [Rosengren (2002)].**

$$X_+ \equiv q^{-1}\frac{q^{\lambda+1}-q^{-(\lambda+1)}}{q-q^{-1}}\beta, \qquad X_- \equiv -q\frac{q^{\rho}-q^{-\rho}}{q-q^{-1}}\gamma, \qquad q^H \equiv q^{\frac{1}{2}(\lambda-\rho)}$$

$\lambda, \rho \longrightarrow -\infty$ but $\lambda - \rho$ fixed

# $\mathfrak{F}_q(\mathfrak{sl}_2)$ as a deformation of $U_q(\mathfrak{sl}_2^*)$ (Greek $\Longrightarrow$ Latin)

## Product

$\mathcal{F}_q(SL_2)$ ✓

$$ba = qab, \qquad ca = qac, \qquad bdq^{-1}db, \qquad cd = q^{-1}dc,$$

$$da - ad = (q - q^{-1})bc, \qquad bc = cb$$

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$$\triangle(a) = a \otimes a + b \otimes c, \qquad \triangle(b) = a \otimes b + b \otimes d,$$
$$\triangle(c) = c \otimes a + d \otimes c, \qquad \triangle(d) = c \otimes b + d \otimes d$$

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### Counit and Antipode

$$\epsilon(a) = 1, \qquad \epsilon(b) = 0, \qquad \epsilon(c) = 0, \qquad \epsilon(d) = 1$$
$$S(a) = d, \qquad S(b) = -qb, \qquad S(c) = -q^{-1}c, \qquad S(d) = a$$

# Duality between $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2^*)$

## Proposition

The duality pairing between $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2^*)$ is given by

$$\langle q^{\pm\frac{H}{2}}, a\rangle = q^{\pm 1}, \qquad \langle q^{\pm\frac{H}{2}}\rangle = q^{\mp\frac{1}{2}}$$
$$\langle X_+, b\rangle = 1, \qquad \langle X_-, c\rangle = 1$$

# The quantum double $D(U_q(\mathfrak{sl}_2))$

## The double construction

$$D(U_q(\mathfrak{sl}_2)) \equiv U_q(\mathfrak{su}_2) \bowtie C_q[SL(2,\mathbb{C})]^{\mathrm{op}} \cong U_q(\mathfrak{su}_2) \bowtie U_q(\mathfrak{su}_2^*)^{\mathrm{op}}$$

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## Crossed products (involving $H$)

$$[q^{\frac{H}{2}}, a] = 0, \qquad q^{\frac{H}{2}} b = q^{-1} b q^{\frac{H}{2}}, \qquad q^{\frac{H}{2}} c = q c q^{\frac{H}{2}}, \qquad [q^{\frac{H}{2}}, d] = 0$$

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## Crossed products (involving $X_\pm$)

$$X_- a = q^{-1} a X_- + b q^{\frac{H}{2}}, \qquad [X_-, b] = 0,$$
$$[X_-, c] = q(q^{\frac{H}{2}} d - q^{-\frac{H}{2}} a), \qquad d X_- = q^{-1} X_- d + q^{\frac{H}{2}} b,$$
$$a X_+ = q X_+ a + q^{-\frac{H}{2}} c, \qquad [X_+, c] = 0,$$
$$[X_+, b] = q^{-1}(q^{\frac{H}{2}} a - q^{-\frac{H}{2}} d), \qquad X_+ d = q d X_+ + c q^{\frac{H}{2}}$$

## The Hopf algebroid $\mathfrak{F}_q(\mathfrak{sl}_2)$ is self-dual [Koelink, Van Norden, Rosengren (2003)]

**Proposition.**

$$\left\langle X_+, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & F(\lambda-1) \\ 0 & 0 \end{pmatrix}, \quad \left\langle X_-, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 0 \\ \frac{1}{F(\lambda)} & 0 \end{pmatrix}$$

$$\left\langle K^+, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} q^{\frac{1}{2}} T_{-1} & 0 \\ 0 & q^{-\frac{1}{2}} T_{+1} \end{pmatrix},$$

$$\lambda, \rho \longrightarrow -\infty$$

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**Conclusion 2**

**The Hopf algebroid $\mathcal{F}_q(\mathfrak{sl}_2)$ could be realized as a deformation/extension of the quantum group $D(U_q(\mathfrak{sl}_2))$.**

# IV. Enlarging the structure of quantum symmetries: $\kappa$-Poincaré

# $\kappa$-Poincaré Hopf algebra (Product and Coproduct)

The $\kappa$-Poincaré Hopf algebra $U(\mathcal{P}_\kappa)$ is generated by the Lorentz generators $N_\mu$ and momentum generators $p_\mu$, such that

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## Product

$$[p_i, p_j] = 0, \qquad [M, N_1] = N_2, \qquad [N_0, N_2] = -N_1, \qquad [N_1, N_2] = -N_0,$$

$$[N_0, p_0] = 0, \qquad [N_0, p_i] = i\epsilon_{ij} p_i, \qquad [N_i, p_0] = -i\epsilon_{ij} p_j e^{-\lambda p_0}$$

$$[N_i, p_j] = \frac{i}{2}\epsilon_{ij} e^{-\lambda p_0} \left( \frac{e^{2\lambda p_0} - 1}{\lambda} - \lambda \vec{p}^2 \right)$$

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## Product

$$\begin{cases} [p_i, p_j] = 0, \quad [M, N_1] = N_2, \quad [N_0, N_2] = -N_1, \quad [N_1, N_2] = -N_0, \\ [N_0, p_0] = 0, \quad [N_0, p_i] = i\epsilon_{ij} p_i, \quad [N_i, p_0] = -i\epsilon_{ij} p_j e^{-\lambda p_0} \\ [N_i, p_j] = \frac{i}{2}\epsilon_{ij} e^{-\lambda p_0} \left( \frac{e^{2\lambda p_0} - 1}{\lambda} - \lambda \vec{p}^2 \right) \end{cases}$$

## Coproduct

$$\begin{cases} \triangle(p_0) = p_0 \otimes 1 + 1 \otimes p_0, \quad \triangle(p_i) = p_i \otimes 1 + e^{\lambda p_0} \otimes p_i \\ \triangle(N_i) = 1 \otimes N_i + N_i \otimes e^{-\lambda p_0} + \lambda N_0 \otimes p_i e^{-\lambda p_0}, \\ \triangle(N_0) = 1 \otimes N_0 + N_0 \otimes 1 \end{cases}$$

# $\kappa$-Poincaré Hopf algebra (Counit and Antipode)

## Counit

$$\epsilon(p_\mu) = \epsilon(N_\mu) = 0$$

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## Counit

$$\epsilon(p_\mu) = \epsilon(N_\mu) = 0$$

## Antipode

$$S(p_0) = -p_0, \qquad S(p_i) = -p_i e^{-\lambda p_0}$$
$$S(N_0) = -N, \qquad S(N_i) = -e^{-\lambda p_0}(N_i + \lambda N p_i)$$

# The Heisenberg Hopf algebroid $\mathcal{H}$

## The Heisenberg algebra $\mathcal{H}$ (+Hopf algebroid structure)

$$[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \qquad [p_\mu, x_\nu] = -i\eta_{\mu\nu}$$

can be equipped with a Hopf algebroid structure via

$$\tilde{\Delta}_0(p_\mu) = p_\mu \otimes 1 + 1 \otimes p_\mu, \qquad \tilde{\Delta}_0(x_\mu) = x_\mu \otimes 1$$

$$\tilde{\epsilon}_0(h) = h \triangleright 1, \qquad \tilde{S}_0(p_\mu) = -p_\mu, \qquad \tilde{S}_0(x_\mu) = -x_\mu,$$

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## Algebra exercise!

We can recover the algebra part of $U(\mathcal{P}_\kappa)$ from $\mathcal{H}$ using

$$M_i = x_i Z^{-1}\left(\frac{Z^2-1}{2\lambda} - \frac{\lambda}{2}\vec{p}^2\right) - x_0 p_i, \qquad M_0 = x_1 p_2 - x_2 p_1$$

with $Z = e^{\lambda p_0}$.

# Recovering $U(\mathcal{P}_\kappa)$ from the Heisenberg Hopf algebroid

**Proposition [Jurić, Meljanac, Strajn, Pachol (2013)]**

The Hopf algebra structure of $U(\mathcal{P}_\kappa)$ could be recover from the Hopf algebroid structure over $\mathcal{H}$ by twisting it with

$$\mathcal{F} = \exp(-i\lambda p_0 \otimes x_k p_k)$$

i.e.

$$\triangle(\cdot) = \mathcal{F}\tilde{\triangle}_0(\cdot)\mathcal{F}, \qquad \epsilon = \tilde{\epsilon}_0 \qquad \text{and} \qquad S(\cdot) = \chi\tilde{S}_0(\cdot)\chi^{-1}$$

where $\chi = \exp(i\lambda p_0 x_i p_i)$.

# Recovering $U(\mathcal{P}_\kappa)$ from the Heisenberg Hopf algebroid

## Proposition [Juríc, Meljanac, Strajn, Pachol (2013)]

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## Conclusion 3

**The Heisenberg Hopf algebroid $\mathcal{H}$ is a (twist) deformation of the quantum group $U(\mathcal{P}_\kappa)$**

## Main References


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# Thank You!

# Any Questions? Please ask!