Hopf heaps

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Heaps [Prüfer '24, Baer '29]

A **heap** is a set A together with a ternary operation

$$[-,-,-] : A \times A \times A \to A,$$

such that for all $a_i \in A$, $i = 1, \ldots, 5$,

(a) $[[a_1, a_2, a_3], a_4, a_5] = [a_1, a_2, [a_3, a_4, a_5]]$,

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- One deduces that $[[a_1, a_2, a_3], a_4, a_5] = [a_1, [a_4, a_3, a_2], a_5]$
- Any (abelian) group is a (-n abelian) heap: $[a, b, c] = a - b + c$.

Homomorphism of heaps: a function $f : A \to B$ such that

$$f[a_1, a_2, a_3] = [f(a_1), f(a_2), f(a_3)].$$

Affine spaces

- In an affine space A over a vector space V:
  (a) any $a, b \in A$ differ by a unique vector $\overrightarrow{ab}$;
  (b) any point can be shifted by a vector to a (unique) point, in particular, for all $a, b, c \in A$,
  $$a + \overrightarrow{bc} \in A;$$
  (c) can shift any pair of points by a rescaled difference between them, i.e., for all $a, b \in A$ and $\lambda \in \mathbb{F}$,
  $$a + \lambda \overrightarrow{ab} \in A.$$
- Observation: we can get rid of V altogether.

Affine spaces

An affine space A is a heap with an $\mathbb{F}$-action (heap of $\mathbb{F}$-modules) $(\lambda, a, b) \mapsto \lambda \triangleright_a b$, such that

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- $[a, b, c] = a + \overrightarrow{bc}$;
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Affine spaces

A morphism of affine spaces (A, V) to (B, W) is a function $f : A \to B$ which induces a linear transformation $\hat{f} : V \to W$ such that

$$\hat{f}\left(\overrightarrow{ab}\right) = \overrightarrow{f(a)f(b)}.$$

This is equivalent to say that f is a morphism of heaps such that

$$f(\lambda \triangleright_a b) = \lambda \triangleright_{f(a)} f(b)$$

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- The assignment:

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- $\mathrm{Tn}(A)$ acts on A freely and transitively
$$c \cdot \tau_a^b = [c, a, b].$$

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- TB & M. Hryniewicka '23: the role of the translation automorphisms made explicit.

Hopf heaps [Grunspan '02]

A **Hopf heap** is a coalgebra C together with a coalgebra map

$$\chi : C \otimes C^{\mathrm{co}} \otimes C \to C, \qquad a \otimes b \otimes c \mapsto [a, b, c],$$

such that for all $a, b, c, d, e \in C$,

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$$f([a, b, c]) = [f(a), f(b), f(c)].$$

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A Hopf algebra H is a Hopf heap with $[a, b, c] = aS(b)c$.

Translations

Let (C, χ) be a Hopf heap.

- For all $a, b \in C$, the linear map

$$\tau_a^b : C \to C, \qquad c \mapsto \chi(c \otimes a \otimes b) = [c, a, b],$$

is called a **right** (a, b)**-translation.** The space spanned by all right (a, b)-translations is denoted by $\mathrm{Tn}(C)$.

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- Symmetrically, linear maps

$$\sigma_b^a : C \to C, \qquad c \mapsto \chi(a \otimes b \otimes c) = [a, b, c],$$

  are called **left** (a, b)**-translations** and the space spanned by all of them is denoted by $\widehat{\mathrm{Tn}}(C)$.

Properties of translations

Let (C, χ) be a Hopf heap. Then, for all $a, b, c, d \in C$,

$$\Delta(\tau_a^b(c)) = \sum \tau_{a_{(2)}}^{b_{(1)}}(c_{(1)}) \otimes \tau_{a_{(1)}}^{b_{(2)}}(c_{(2)}),$$

$$\sum \tau_{a_{(1)}}^{[a_{(2)},b,c]} = \varepsilon(a)\tau_b^c,$$

$$\sum \tau_{a_{(1)}}^{a_{(2)}} = \varepsilon(a)\mathrm{id},$$

$$\tau_c^d \circ \tau_a^b = \tau_a^{[b,c,d]}.$$

In addition if the Grunspan map ϑ exists, then

$$\sum \tau_{a_{(2)}}^{[\vartheta(a_{(1)}),b,c]} = \varepsilon(a)\tau_b^c,$$

$$\sum \tau_{a_{(2)}}^{\vartheta(a_{(1)})} = \varepsilon(a)\mathrm{id},$$

$$\tau_c^d \circ \tau_a^{\vartheta(b)} = \tau_{[c,b,a]}^d.$$

Theorem (translation Hopf algebras)

- $\mathrm{Tn}(C)$ is a bialgebra wrt the opposite composition, and :

$$\Delta(\tau_a^b) = \sum \tau_{a_{(2)}}^{b_{(1)}} \otimes \tau_{a_{(1)}}^{b_{(2)}}, \qquad \varepsilon(\tau_a^b) = \varepsilon(a)\varepsilon(b).$$

- If ϑ exists, then $\mathrm{Tn}(C)$ is a Hopf algebra with the antipode

$$S(\tau_a^b) = \tau_b^{\vartheta(a)}.$$

- If $f : C \to D$ is a morphism of Hopf heaps, then

$$\mathrm{Tn}(f) : \mathrm{Tn}(C) \to \mathrm{Tn}(D), \qquad \tau_a^b \mapsto \tau_{f(a)}^{f(b)},$$

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- Similar statements hold for $\widehat{\mathrm{Tn}}(C)$.
- For all grouplike $x \in C$, C with

$$1 = x/\varepsilon(x), \qquad ab = [a, x, b], \qquad S(a) = [x, a, x].$$

is a Hopf algebra isomorphic to $\mathrm{Tn}(C)$ and $\widehat{\mathrm{Tn}}(C)$.

Galois co-objects

Definition

A right H-module coalgebra C is a **right Hopf-Galois co-object** if

(a) $\ker \varepsilon = \mathbb{F}\langle c \cdot h - c\varepsilon(h) \mid c \in C, h \in H \rangle$,

(b) the *canonical map*

$$\text{can} : C \otimes H \to C \otimes C, \qquad c \otimes h \mapsto \sum c_{(1)} \otimes c_{(2)} \cdot h,$$

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A **left Hopf-Galois co-object** is defined symmetrically.
A coalgebra C that is both a right and left Hopf-Galois co-object of Hopf algebras whose actions on C commute is called a **bi-Galois co-object**.

The Ehresman Hopf algebra

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- The subspace

$$I = \mathbb{F}\langle a \otimes b\varepsilon(c) - \sum a \cdot \tau(b \otimes c_{(1)}) \otimes c_{(2)} \mid a, b, c \in C \rangle \subseteq C \otimes C,$$

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- $\mathrm{E}(C, H) := C^{\mathrm{co}} \otimes C/I$ is a Hopf algebra

$$1 = \overline{\sum e_{(1)} \otimes e_{(2)}}, \quad \overline{a \otimes b}\,\overline{c \otimes d} = \overline{a \cdot \tau(b \otimes c) \otimes d},$$

$$S(\overline{a \otimes b}) = \overline{\sum a \cdot \tau(b \otimes e_{(1)}) \otimes e_{(2)}},$$

  where $e \in \varepsilon^{-1}(1)$.

Theorem (heaps to Galois co-objects)

Let (C, χ) be a Hopf heap. Then:

- C is a right Hopf-Galois co-object over the right translation Hopf algebra $\mathrm{Tn}(C)$ with the action,

$$c \cdot \tau_a^b = \tau_a^b(c) = [c, a, b].$$

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- $\mathrm{Tn}(C)$ and $\widehat{\mathrm{Tn}}(C)$ are Hopf algebras.

Theorem (Galois co-objects to heaps)

Let H be a Hopf algebra and C be a right H-Hopf-Galois co-object. Then

- C is a Hopf heap with the Grunspan map by the operation

$$\chi_{(C,H)} : C \otimes C^{\mathrm{co}} \otimes C \to C, \qquad a \otimes b \otimes c \mapsto a \cdot \tau(b \otimes c),$$

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- $H \cong \mathrm{Tn}(C)$ as Hopf algebras.

Equivalence of categories

- A morphism of Galois co-objects (C, H) to (D, K) is a pair (f, g)
  (a) $f : C \to D$ is a homomorphism of coalgebras,
  (b) $g : H \to K$ is a homomorphism of Hopf algebras,
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$$f(c \cdot h) = f(c) \cdot g(h).$$

- The functors

$$\begin{aligned} \mathrm{Ga} :& \mathcal{HH} \to \mathcal{HG}, \qquad (C, \chi) \mapsto (C, \mathrm{Tn}(C)), \quad f \mapsto (f, \mathrm{Tn}(f)), \\ \mathrm{He} :& \mathcal{HG} \to \mathcal{HH}, \qquad (C, H) \mapsto (C, \chi_{(C,H)}), \quad (f, g) \mapsto f, \end{aligned}$$

are a pair of inverse equivalences between categories of Hopf heaps and right Hopf-Galois co-objects.

References

- The talk is based on:
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- Hopf heaps or (co)torsors are studied in:

  - C. Grunspan, Quantum torsors, *JPAA* **184** (2003), 229–255.

  - P. Schauenburg, Quantum torsors and Hopf-Galois objects, arXiv:math/0208047 (2002); Quantum torsors with fewer axioms, arXiv:math/0302003 (2003)

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