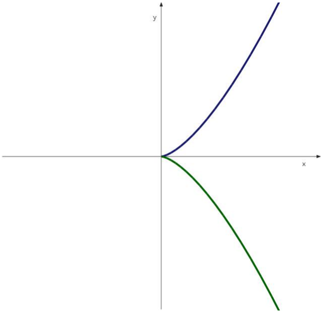
Differential operators on the cusp

Ulrich Krähmer and Myriam Mahaman



Part 1: Definitions

The field k, the algebraic set X, and the main result

1. Let k be an algebraically closed field of characteristic 0 and

$$X = \{(\lambda_1, \ldots, \lambda_d) \in k^d \mid r_1(\lambda_1, \ldots, \lambda_d) = \ldots = r_n(\lambda_1, \ldots, \lambda_d) = 0\}$$

be the algebraic set defined by $r_1, \ldots, r_n \in k[x_1, \ldots, x_d]$.

The field k, the algebraic set X, and the main result

1. Let k be an algebraically closed field of characteristic 0 and
$$X = \{(\lambda_1, \ldots, \lambda_d) \in k^d \mid r_1(\lambda_1, \ldots, \lambda_d) = \ldots = r_n(\lambda_1, \ldots, \lambda_d) = 0\}$$
be the algebraic set defined by $r_1, \ldots, r_n \in k[x_1, \ldots, x_d]$.
2. Ultimately I will only speak about the example of the **cusp**
$$\{(\lambda_1, \lambda_2) \in k^2 \mid \lambda_1^3 - \lambda_2^2 = 0\}$$
from the title page and explain what the following means, and why Myriam and I think it is plausible, true, interesting, and nontrivial:

Theorem (K., Mahaman 2023)

The differential operators on the cusp form an involutive Hopf algebroid.

The commutative k-algebras A and $k[X]$

1. Throughout, A is a commutative k-algebra.
2. The guiding example is the **coordinate ring** of X,

$$k[X] := k[x_1, \ldots, x_d]/\sqrt{\langle r_1, \ldots, r_n \rangle}.$$

   Here $\langle S \rangle \lhd R$ denotes the ideal generated by a subset S of a (unital associative) ring R and $\sqrt{I} = \{a \in R \mid \exists l : a^l \in I\}$ is the radical of an ideal $I \lhd R$ in a commutative ring R.
3. Hilbert's Nullstellensatz identifies the elements $f \in k[X]$ with the **regular functions** $X \to k$; the generators x_i are the coordinates

$$x_i \colon X \to k, \quad (\lambda_1, \ldots, \lambda_d) \mapsto \lambda_i.$$

The field $k(X)$

1. From now on we assume that X is an **affine variety** (is irreducible), that is, that $k[X]$ is an integral domain.
2. For a plane curve ($d = 2, n = 1$) this means that the defining polynomial $r_1 \in k[x_1, x_2]$ is irreducible, as is the case for the cusp.

The field $k(X)$

1. From now on we assume that X is an **affine variety** (is irreducible), that is, that $k[X]$ is an integral domain.
2. For a plane curve $(d = 2, n = 1)$ this means that the defining polynomial $r_1 \in k[x_1, x_2]$ is irreducible, as is the case for the cusp.
3. The field of **rational functions** $X \to k$ is the quotient field

$$k(X) := \{\frac{f}{g} \mid f, g \in k[X], g \neq 0\}.$$

The field $k(X)$

1. From now on we assume that X is an **affine variety** (is irreducible), that is, that $k[X]$ is an integral domain.
2. For a plane curve ($d = 2, n = 1$) this means that the defining polynomial $r_1 \in k[x_1, x_2]$ is irreducible, as is the case for the cusp.
3. The field of **rational functions** $X \to k$ is the quotient field

$$k(X) := \{\frac{f}{g} \mid f, g \in k[X], g \neq 0\}.$$

4. For the cusp, we have a k-algebra isomorphism

$$k(X) \cong k(t) = \{\frac{f}{g} \mid f, g \in k[t], g \neq 0\}, \quad x_1 \mapsto t^2, \quad x_2 \mapsto t^3.$$

So the cusp is birationally isomorphic to the affine line k.

The affine variety $\bar{X}$

1. The **normalisation** of $k[X]$ is its integral closure $\overline{k[X]}$ in $k(X)$, that is, the k-algebra of all roots $r \in k(X)$ of monic polynomials

$$r^m + f_1 r^{m-1} + \cdots + f_{m-1} r + f_m = 0, \quad f_j \in k[X]$$

with coefficients in $k[X]$.

The affine variety $\bar{X}$

1. The **normalisation** of $k[X]$ is its integral closure $\overline{k[X]}$ in $k(X)$, that is, the k-algebra of all roots $r \in k(X)$ of monic polynomials

$$r^m + f_1 r^{m-1} + \cdots + f_{m-1} r + f_m = 0, \quad f_j \in k[X]$$

with coefficients in $k[X]$.

Theorem (Noether 1926)

$\overline{k[X]}$ is a finitely generated $k[X]$-module.

2. In particular, it is a finitely generated k-algebra, hence by the Nullstellensatz the coordinate ring of an affine variety $\bar{X}$.

The morphism $\pi \colon \bar{X} \to X$

1. Furthermore, the inclusion

$$k[X] \hookrightarrow \overline{k[X]}$$

corresponds to a surjection with finite fibres

$$\pi \colon \bar{X} \twoheadrightarrow X, \qquad 0 < |\pi^{-1}(x)| < \infty \ \forall x \in X.$$

The morphism $\pi\colon \bar{X} \to X$

1. Furthermore, the inclusion

$$k[X] \hookrightarrow \overline{k[X]}$$

corresponds to a surjection with finite fibres

$$\pi\colon \bar{X} \twoheadrightarrow X, \qquad 0 < |\pi^{-1}(x)| < \infty \ \forall x \in X.$$

2. If X is nonsingular (smooth), then $X = \bar{X}$. For a curve, this is iff.

The morphism $\pi \colon \bar{X} \to X$

1. Furthermore, the inclusion
$$k[X] \hookrightarrow \overline{k[X]}$$
corresponds to a surjection with finite fibres
$$\pi \colon \bar{X} \twoheadrightarrow X, \qquad 0 < |\pi^{-1}(x)| < \infty \ \forall x \in X.$$
2. If X is nonsingular (smooth), then $X = \bar{X}$. For a curve, this is iff.
3. The cusp is singular, but π is bijective: the normalisation of $k[t^2, t^3] \subseteq k(t)$ is $k[t] \subseteq k(t)$, so $\bar{X} = k$ is the affine line, and
$$\pi \colon k = \bar{X} \to X, \quad \tau \mapsto (\tau^2, \tau^3).$$

The bialgebroid H

1. We'll study (left) bialgebroids H over A with source $=$ target and injective anchor $\hat{\varepsilon}$; I'll suppress them and assume $A \subseteq H \subseteq \mathrm{End}_k(A)$.

$$s = t \colon A \hookrightarrow H, \quad \hat{\varepsilon} \colon H \hookrightarrow \mathrm{End}_k(A).$$

The bialgebroid H

1. We'll study (left) bialgebroids H over A with source = target and injective anchor $\hat{\varepsilon}$; I'll suppress them and assume $A \subseteq H \subseteq \mathrm{End}_k(A)$.

$$s = t\colon A \hookrightarrow H, \quad \hat{\varepsilon}\colon H \hookrightarrow \mathrm{End}_k(A).$$

2. The coproduct will be denoted by

$$\Delta\colon H \to H \times_A H, \quad h \mapsto h_{(1)} \otimes_A h_{(2)}.$$

3. For $a \in A$, $\sum_i g_i \otimes_A h_i \in H \times_A H$, we have

$$\sum_i ag_i \otimes_A h_i = \sum_i g_i \otimes_A ah_i \quad \text{and even} \quad \sum_i g_i a \otimes_A h_i = \sum_i g_i \otimes_A h_i a$$

because we are in $H \otimes_A H$ and even in $H \times_A H \subseteq H \otimes_A H$.

The involutive Hopf algebroid H

Definition

H becomes an **involutive Hopf algebroid** if we can and do choose a morphism of k-algebras $S\colon H \to H^{\mathrm{op}}$ satisfying for $a \in A, h \in H$

$$S^2(h) = h, \qquad S(a) = a, \qquad S(h_{(1)})h_{(2)} = [S(h)](1),$$

$$(\Delta \otimes_A \mathrm{id}_H) \circ \Delta' = (\Delta' \otimes_A \mathrm{id}_H) \circ \Delta,$$

where $\Delta'(S(h)) = S(h_{(2)}) \otimes_A S(h_{(1)})$.

The A-ring $\mathcal{D}(A)$

1. The inclusion $A \subseteq \mathrm{End}_k(A)$ identifies the elements $a \in A$ with the multiplication operators $=:$ differential operators of order 0

$$A \to A, \quad b \mapsto ab.$$

Higher order differential operators are defined inductively:

The A-ring $\mathcal{D}(A)$

1. The inclusion $A \subseteq \mathrm{End}_k(A)$ identifies the elements $a \in A$ with the multiplication operators $=:$ differential operators of order 0

$$A \to A, \quad b \mapsto ab.$$

Higher order differential operators are defined inductively:

Definition

The A-ring $\mathcal{D}(A)$ of k-**linear differential operators** over A is the filtered k-subalgebra $\mathcal{D}(A) = \bigcup_{n \in \mathbb{N}} \mathcal{D}(A)^n \subseteq \mathrm{End}_k(A)$, where

1. $\mathcal{D}(A)^0 = A$,
2. $\mathcal{D}(A)^n = \{D \in \mathrm{End}_k(A) \mid Da - aD \in \mathcal{D}(A)^{n-1} \, \forall a \in A\}$, $n \geqslant 1$.

Part 2: Motivations

The Nakai conjecture

1. For all A, we have an isomorphism of A-modules

$$\mathcal{D}(A)^1 \to \mathrm{Der}_k(A) \oplus A, \quad D \mapsto (D - D(1), D(1)).$$

The Nakai conjecture

1. For all A, we have an isomorphism of A-modules

$$\mathcal{D}(A)^1 \to \mathrm{Der}_k(A) \oplus A, \quad D \mapsto (D - D(1), D(1)).$$

Theorem (Grothendieck 1967, Sweedler 1974)

If X is smooth, this induces an iso $\mathcal{D}(k[X]) \cong U(k[X], \mathrm{Der}_k(k[X]))$.

Here the right hand side is the universal eneloping algebra of the Lie-Rinehart algebra $(k[X], \mathrm{Der}_k(k[X]))$.

The Nakai conjecture

1. For all A, we have an isomorphism of A-modules

$$\mathcal{D}(A)^1 \to \mathrm{Der}_k(A) \oplus A, \quad D \mapsto (D - D(1), D(1)).$$

Theorem (Grothendieck 1967, Sweedler 1974)

If X is smooth, this induces an iso $\mathcal{D}(k[X]) \cong U(k[X], \mathrm{Der}_k(k[X]))$.

Here the right hand side is the universal eneloping algebra of the Lie-Rinehart algebra $(k[X], \mathrm{Der}_k(k[X]))$.

Conjecture (Nakai 1961, sort of)

This is an if and only if ("but nothing is yet known about it").

By now, it is known for curves and a few more examples.

The Zariski-Lipman conjecture

1. If I is the kernel of the multiplication map $k[X] \otimes_k k[X] \to k[X]$ and $\Omega^1(X) = I/I^2$ is the $k[X]$-module of Kähler differentials, then $\mathrm{Der}_k(k[X]) \cong \mathrm{Hom}_A(\Omega^1(X), k[X])$ and we have:

Theorem

X is smooth iff $\Omega^1(X)$ is a projective $k[X]$-module of rank $\dim(X)$.

2. In particular: If X is smooth, then $\mathrm{Der}_k(k[X])$ is a finitely generated projective $k[X]$-module.

The Zariski-Lipman conjecture

1. If I is the kernel of the multiplication map $k[X] \otimes_k k[X] \to k[X]$ and $\Omega^1(X) = I/I^2$ is the $k[X]$-module of Kähler differentials, then $\mathrm{Der}_k(k[X]) \cong \mathrm{Hom}_A(\Omega^1(X), k[X])$ and we have:

Theorem

X is smooth iff $\Omega^1(X)$ is a projective $k[X]$-module of rank $\dim(X)$.

2. In particular: If X is smooth, then $\mathrm{Der}_k(k[X])$ is a finitely generated projective $k[X]$-module. The Nakai conjecture would imply:

Conjecture (Zariski, Lipman 1965)

This is an if and only if.

The Poincaré-Birkhoff-Witt theorem

1. Recall furthermore that Rinehart has extended the Poincaré-Birkhoff-Witt theorem to Lie-Rinehart algebras:

Theorem (Rinehart 1963)

If (A, L) is a Lie-Rinehart algebra and L is a projective A-module, then

$$\mathrm{Gr}(U(A, L)) \cong S_A L.$$

Here Gr is the associated graded k-algebra, where $U(A, L)$ is filtered with $U(A, L)^n$ being the A-module generated by all monomials $X_1 \cdots X_l$, $l \leqslant n$, $X_j \in L$, and $S_A L$ is the symmetric algebra of the A-module L.

The motivation for our theorem

1. $U(A, L)$ is for all Lie-Rnehart algebras a left Hopf algberoid. It is not always a full or even involutive Hopf algebroid.
2. The obvious extension of Cartier-Milnor-Moore holds in the projective case:

Theorem (Moerdijk, Mrčun 2010)

The cocommutative conilpotent left Hopf algberoids H that are projective as A-modules are precisely those of the form $U(A, L)$.

3. Hence we think it is natural to ask:

Question

For which A is $\mathcal{D}(A)$ what sort of Hopf algebroid?

Part 3: The main result – details

The formulas: generators

1. Abbreviate from now on $A := k[t^2, t^3] \subseteq B := k[t, t^{-1}]$ and

$$\partial := \frac{d}{dt} \colon B \to B, \quad t^j \mapsto jt^{j-1}.$$

2. The following are differential operators of A:

$$D_0 := t\partial,\ D_1 := t^2\partial \in \mathcal{D}(A)^1,$$

$$E_{-1} := t\partial^2 - \partial,\ E_{-2} := \partial^2 - \frac{2}{t}\partial \in \mathcal{D}(A)^2,$$

$$E_{-3} := \partial^3 - \frac{3}{t}\partial^2 + \frac{3}{t^2}\partial \in \mathcal{D}(A)^3.$$

The formulas: relations

Proposition (Smith 1981)

The ring $\mathcal{D}(A)$ is generated as an algebra over k by the elements $x, y, D_0, E_{-2}, E_{-3}$, satisfying the following relations:

$$[x, y] = 0, \quad x^3 = y^2, \quad [E_{-2}, E_{-3}] = 0, \quad E_{-2}^3 = E_{-3}^2,$$
$$xE_{-2} = D_0(D_0 - 3), \quad E_{-2}x = (D_0 + 2)(D_0 - 1), \quad yE_{-2} = D_1(D_0 - 3),$$
$$E_{-2}y = D_1(D_0 + 3), \quad xE_{-3} = E_{-1}(D_0 - 4), \quad E_{-3}x = E_{-1}(D_0 + 2),$$
$$yE_{-3} = D_0(D_0 - 2)(D_0 - 4), \quad E_{-3}y = (D_0 + 3)(D_0 + 1)(D_0 - 1),$$
$$[D_0, x] = 2x, \ [D_0, y] = 3y, \ [D_0, E_{-2}] = -2E_{-2}, \ [D_0, E_{-3}] = -3E_{-3},$$

where $D_1 = y(D_0 - 1)E_{-2} - x^2E_{-3}$ *and* $E_{-1} = x(D_0 - 1)E_{-3} - yE_{-2}^2$.

The formulas: Δ and S

1. $\Delta : \mathcal{D}(A) \to \mathcal{D}(A) \times_A \mathcal{D}(A)$ is the morphism of A-rings such that

$$\Delta(D_0) = D_0 \otimes_A 1 + 1 \otimes_A D_0,$$

$$\Delta(E_{-2}) = E_{-2} \otimes_A 1 + 2D_0 \otimes_A (D_0 - 1)E_{-2} - 2D_1 \otimes_A E_{-3} + 1 \otimes_A E_{-2},$$

$$\begin{aligned}\Delta(E_{-3}) = {} & E_{-3} \otimes_A 1 + 3E_{-2} \otimes_A E_{-1} - 3E_{-1} \otimes_A E_{-2} \\ & + 6D_0 \otimes_A (D_0 - 1)E_{-3} - 6D_1 \otimes_A E_{-2}^2 + 1 \otimes_A E_{-3},\end{aligned}$$

2. $S : \mathcal{D}(A) \to \mathcal{D}(A)^{\mathrm{op}}$ is the involutive A-ring morphism such that

$$S(D_0) = 1 - D_0, \quad S(E_{-2}) = E_{-2}, \quad S(E_{-3}) = -E_{-3}.$$

Part 4: The main result – ingredients in the proof

Differential operators on curves

1. Assume X is a curve. Here is beautiful stuff:

Theorem (Smith, Stafford 1988)

$\mathcal{D}(k[X])$ is a finitely generated and Noetherian k-algebra with a unique minimal ideal $J \lhd \mathcal{D}(A)$, and $\dim_k(\mathcal{D}(A)/J) < \infty$. Tfae:

- *π is injective.*
- *$\mathcal{D}(k[X])$ is a simple ring.*
- *$\mathcal{D}(k[X])$ and $\mathcal{D}(k[\bar{X}])$ are Morita equivalent.*
- *$\mathrm{Gr}(\mathcal{D}(k[X]))$ is Noetherian.*
- *The global dimension of $\mathcal{D}(k[X])$ is 1.*

The grading

1. All we took from that paper is that $\mathcal{D}(k[X])$ embeds into $\mathcal{D}(k(X))$ and in fact for the cusp into $\mathcal{D}(B)$ as the k-subalgebra of those $D \in \mathcal{D}(B)$ that map $A = k[t^2, t^3] \subseteq B = k[t, t^{-1}]$ to itself.

The grading

1. All we took from that paper is that $\mathcal{D}(k[X])$ embeds into $\mathcal{D}(k(X))$ and in fact for the cusp into $\mathcal{D}(B)$ as the k-subalgebra of those $D \in \mathcal{D}(B)$ that map $A = k[t^2, t^3] \subseteq B = k[t, t^{-1}]$ to itself.
2. This yields a grading on the right $k[D_0]$-module $\mathcal{D}(A)$:

Proposition

The right $k[D_0]$-module $\mathcal{D}(A)$ is free with a basis $\{\ldots, E_{-3}, E_{-2}, E_{-1}, E_0 = 1, D_1, t^2, t^3, \ldots\}$ described on the next slide.

The weird polynomials

1. The operators $E_{-d} \colon A \to A$ are given by
$$E_{-d}(t^j) = \ell_{-d}(j) t^{j-d},$$
where $\ell_{-d} \in \mathbb{Z}[j]$ is such that you get $E_{-d}(t^j) = 0$ if $t^{j-d} \notin k[t^2, t^3]$:
$$\ell_{-d}(j) = \prod_{i \in M} (j - i)$$
with M the set of those $i \in 2\mathbb{N} + 3\mathbb{N}$ such that $i - d < 0$ (so $i \leqslant d - 1$) or $i - d = 1$ (so $i = d + 1$).
2. Remark: $D_0 E_{-d} = E_{-d}(D_0 - d)$, so the basis is also a basis of the left $k[D_0]$-module $\mathcal{D}(A)$.

The filtration that Myriam built

1. The trouble is that the A-modules $\mathcal{D}(A)^n$ of differential operators of order $\leqslant n$ are not projective. However, the A-modules

$$\mathcal{F}_n := \operatorname{span}_A\{D_0 E_{-n}, E_{-n-1}, E_{-n}, \dots, E_{-3}, E_{-2}, E_0\}$$

are free (with the listed elements as basis).

2. We have $\mathcal{F}_n \subseteq \mathcal{F}_{n+2}$ and $\mathcal{D}(A)^{n-1} \subsetneq \mathcal{F}_n \subsetneq \mathcal{D}(A)^{n+1}$.

What for?

1. It is straightforward to show that

$$\mathcal{F}_n \otimes_A \mathcal{F}_n \to \mathrm{Hom}_k(A \otimes_k A, A), \quad D \otimes_A E \mapsto (a \otimes_k b \mapsto D(a)E(b))$$

   is injective, and to use this in order to show that $\mathcal{D}(A) \otimes_A \mathcal{D}(A)$ embeds into $\mathrm{Hom}_k(A \otimes_k A, A)$ as well; from here, we obtain that $\mathcal{D}(A) \times_A \mathcal{D}(A)$ embeds into $\mathcal{D}(B) \times_B \mathcal{D}(B)$, where $B = k[t, t^{-1}]$, and use this to find the Hopf algebroid structure on $\mathcal{D}(A)$.
2. Remark:The filtration $\mathcal{F}_n$ also yields a direct proof of:

Proposition (Ben-Zvi, Nevins 2004)

$\mathcal{D}(A)$ is a flat A-module.

Further reading

1. The observations of Sweedler and Heyneman are all contained in Sweedler's big 1974 article in the memory of Rinehart, see in particular Theorem 18.2 therein.
2. The book by McConnell and Robson ends with a quite good introduction to rings of differential operators.
3. In the early 2000s Saito and Traves extended some of the above to $\mathcal{D}(A)$ where A is an abelian semigroup algebra (in our case $2\mathbb{N} + 3\mathbb{N}$).