Infinitesimal braidings and pre-Cartier categories

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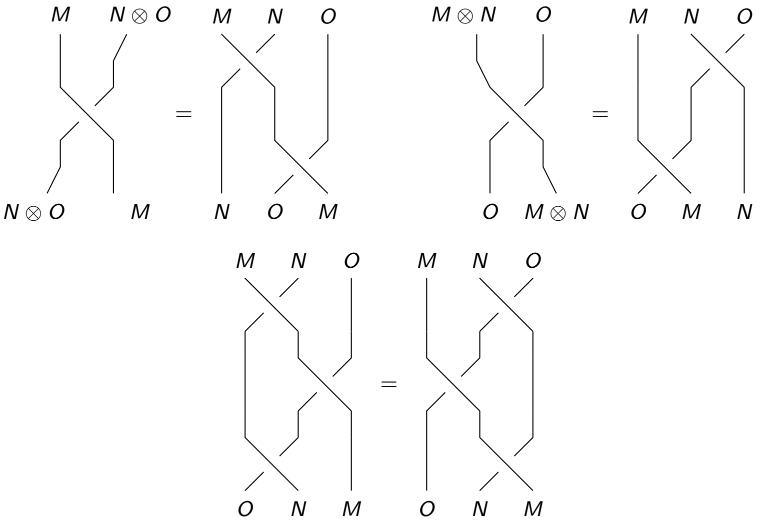
Braided monoidal categories

Category $\mathcal{C}$ with a monoidal structure $\otimes$, the "tensor product", and a natural isomorphism $\sigma_{M,N}\colon M \otimes N \to N \otimes M$ such that



Quasitriangular bialgebras and universal $\mathcal{R}$-matrices

Given a bialgebra (H, Δ, ε) its category of representations (let's say left H-modules) ${}_H\mathcal{M}$ is monoidal with

$$h \cdot (m \otimes n) := \Delta(h)(m \otimes n)$$

for $h \in H$, $m \in M$, $n \in N$, where $M, N \in {}_H\mathcal{M}$.

Theorem (Drinfel'd-Majid '90)

$({}_H\mathcal{M}, \otimes)$ is braided if and only if H is **quasitriangular**, *i.e. $\exists\, \mathcal{R} \in H \otimes H$ invertible s.t.*

$$(\Delta \otimes \mathrm{id}_H)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id}_H \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$$

and $\Delta^{\mathrm{op}}(\cdot) = \mathcal{R}\Delta(\cdot)\mathcal{R}^{-1}$. Then $\sigma^{\mathcal{R}}_{M,N}(m \otimes n) := \mathcal{R}^{\mathrm{op}} \cdot (n \otimes m)$.

Note: σ is **symmetric**, i.e. $\sigma^2 = \mathrm{id}$ if and only if $\mathcal{R}$ is **triangular**, i.e. $\mathcal{R}^{-1} = \mathcal{R}^{\mathrm{op}}$.

Example

i.) Every cocommutative (i.e. $\Delta = \Delta^{\mathrm{op}}$) Hopf algebra is triangular with $\mathcal{R} = 1 \otimes 1$. For example $\Bbbk[G]$ for any group G and $U\mathfrak{g}$ for any Lie algebra $\mathfrak{g}$.

ii.) Group algebra $\mathbb{C}[\mathbb{Z}_n]$ n-cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Triangular structure $\mathcal{R} = \frac{1}{n}\sum_{j,k=0}^{n-1} e^{-\frac{2\pi \mathrm{i} jk}{n}} g^j \otimes g^k$. For $n = 2$: "supergeometry braiding".

iii.) The Drinfel'd double $D(H) = H^* \otimes H$ of a finite-dim. Hopf algebra H is quasitriangular w.r.t. $\mathcal{R} = (1 \otimes e_i) \otimes (e^i \otimes 1)$.

$\tilde{\mathcal{R}} \in (H \otimes H)[[\hbar]] \cong \tilde{H}\hat{\otimes}\tilde{H}$ quasitr. structure on topological bialgebra $\tilde{H} = H[[\hbar]]$.
$\Rightarrow \tilde{R} = \mathcal{R} + \mathcal{O}(\hbar)$ gives quasitriangular bialgebra $(H, \mathcal{R})$ and we can write

$$\tilde{R} = \mathcal{R}(1 \otimes 1 + \hbar\chi + \mathcal{O}(\hbar^2)).$$

What are the properties of $\chi \in H \otimes H$?

Definition (Infinitesimal $\mathcal{R}$-matrix and pre-Cartier bialgebra $(H, \mathcal{R}, \chi)$)

means $(H, \mathcal{R})$ quasitriangular and $\chi \in H \otimes H$ s.t. $\chi\Delta(\cdot) = \Delta(\cdot)\chi$ and

$$(\mathrm{id}_H \otimes \Delta)(\chi) = \chi_{12} + \mathcal{R}_{12}^{-1}\chi_{13}\mathcal{R}_{12}, \qquad (\Delta \otimes \mathrm{id}_H)(\chi) = \chi_{23} + \mathcal{R}_{23}^{-1}\chi_{13}\mathcal{R}_{23}.$$

If also $\mathcal{R}\chi = \chi^{\mathrm{op}}\mathcal{R}$, we call $(H, \mathcal{R}, \chi)$ Cartier bialgebra.

Proposition

$(H, \mathcal{R}, \chi)$ is pre-Cartier bialgebra iff there is a natural transformation $t\colon M \otimes N \to M \otimes N$ on ${}_H\mathcal{M}$ such that

$$t_{M,N\otimes L} = t_{M,N} \otimes L + (\sigma_{M,N}^{-1} \otimes L)(N \otimes t_{M,L})(\sigma_{M,N} \otimes L)$$
$$t_{M\otimes N,L} = M \otimes t_{N,L} + (M \otimes \sigma_{N,L}^{-1})(t_{M,L} \otimes N)(M \otimes \sigma_{N,L}).$$

Moreover, $\mathcal{R}\chi = \chi^{\mathrm{op}}\mathcal{R}$ iff $\sigma_{M,N} \circ t_{M,N} = t_{N,M} \circ \sigma_{M,N}$.

We call braided categories $(\mathcal{C}, \sigma)$ with such transformation pre-Cartier (resp. Cartier).

Some comments on the definition...

- For σ a symmetric braiding with $\sigma \circ t = t \circ \sigma$ we recover the known notion of **Cartier** category. [Cartier '93, Kassel '95, Heckenberger-Vendramin '22]
- The infinitesimal $\mathcal{R}$-matrices χ of $(H, \mathcal{R})$ form a vector space.
- If H is commutative it follows that χ is an inf. $\mathcal{R}$-matrix iff $\chi \in P(H) \otimes P(H)$ and $(H, \mathcal{R}, \chi)$ is Cartier if furthermore $\chi^{\mathrm{op}} = \chi$.

Example

If $(\mathfrak{g}, [\cdot, \cdot], r)$ is a **quasitriangular Lie bialgebra**,
then $\chi := r + r^{\mathrm{op}}$ is an infinitesimal $\mathcal{R}$-matrix of $(U\mathfrak{g}, 1 \otimes 1)$.

Example (Sweedler's 4-dim Hopf algebra)

Generators g, x with $g^2 = 1$, $x^2 = 0$, $xg = -gx$, $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$.
Triangular structure $\mathcal{R}_0 = \frac{1}{2}(1 \otimes 1 + g \otimes 1 + 1 \otimes g - g \otimes g)$.
Note: $\exists$ exhausting 1-parameter family $\mathcal{R}_\lambda$ of triangular structures.

$\exists$ exhausting 1-parameter family $\chi_\alpha := \alpha x g \otimes x$ of infinitesimal $\mathcal{R}$-matrices.
They are Cartier iff $\alpha = 0$.

Induced structure

Proposition

If $(H,\mathcal{R},\chi)$ is (pre-)Cartier bialgebra and $\mathcal{F}\in H\otimes H$ a Drinfel'd twist, then $(H_{\mathcal{F}},\mathcal{F}^{\mathrm{op}}\mathcal{R}\mathcal{F}^{-1},\mathcal{F}\chi\mathcal{F}^{-1})$ is (pre-)Cartier bialgebra.

Example (Sweedler's 4-dim Hopf algebra)

There is a 1-parameter family of Drinfel'd twist $\mathcal{F}_t := 1\otimes 1+\frac{t}{2}xg\otimes x$. We have

$$(\mathcal{R}_\lambda)_{\mathcal{F}_t}=\mathcal{R}_\lambda+\frac{t}{2}(x\otimes xg+x\otimes x+xg\otimes xg-xg\otimes x)$$
$$(\chi_\alpha)_{\mathcal{F}_t}=\chi_\alpha$$

Proposition

Let $f\colon H\to H'$ be a bialgebra map. If $(H,\mathcal{R},\chi)$ is (pre-)Cartier (quasi)triangular, then so is $(f(H),(f\otimes f)(\mathcal{R}),(f\otimes f)(\chi))$.

Corollary

$(\mathcal{R},\chi)$ descends to any bialgebra quotient H/I.

The role of Hochschild cohomology

Theorem (Ardizzoni-Bottegoni-Sciandra-TW '23)

Let $(H, \mathcal{R}, \chi)$ be a pre-Cartier quasitriangular bialgebra. Then

i.) χ *is a* **Hochschild 2-cocycle**, *i.e.* $\chi_{12} + (\Delta \otimes \mathrm{id})(\chi) = \chi_{23} + (\mathrm{id} \otimes \Delta)(\chi)$.

ii.) χ *satisfies the* **infinitesimal quantum Yang-Baxter equation**

$$\begin{aligned}\mathcal{R}_{12}\chi_{12}\mathcal{R}_{13}\mathcal{R}_{23} + \mathcal{R}_{12}\mathcal{R}_{13}\chi_{13}\mathcal{R}_{23} + \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}\chi_{23} \\ = \mathcal{R}_{23}\chi_{23}\mathcal{R}_{13}\mathcal{R}_{12} + \mathcal{R}_{23}\mathcal{R}_{13}\chi_{13}\mathcal{R}_{12} + \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\chi_{12}.\end{aligned}$$

This generalizes a result of [Majid '97] for $(U\mathfrak{g}, 1 \otimes 1)$ with $(\mathfrak{g}, [\cdot,\cdot], r)$ quasitriangular Lie bialgebras as before.

If $(H, \mathcal{R}, \chi)$ is a pre-Cartier quasitriangular **Hopf** algebra, define $\gamma := S(\chi^i)\chi_i \in H$, the **Casimir element**, where $\chi = \chi^i \otimes \chi_i$. It follows that γ is central!

Proposition (Ardizzoni-Bottegoni-Sciandra-TW '23)

If $(H, \mathcal{R}, \chi)$ is a **Cartier triangular Hopf** *algebra, then $\chi = b^1(\frac{\gamma}{2})$ is a Hochschild 2-coboundary, where $b^1(h) := 1 \otimes h - \Delta(h) + h \otimes 1$.*

Dually, the corepresentation category (let's say right comodules) $(\mathcal{M}^H, \otimes)$ of a bialgebra H is braided iff there is a **universal $\mathcal{R}$-form** $\mathcal{R}\colon H \otimes H \to \Bbbk$.

In this case $\sigma_{M,N}(m \otimes n) = n_0 \otimes m_0 \mathcal{R}(m_1 \otimes n_1)$, where $\Delta_M(m) =: m_0 \otimes m_1$, etc.

We define **infinitesimal $\mathcal{R}$-forms** as the algebraic structures on $(H, \mathcal{R})$ s.t. $(\mathcal{M}^H, \otimes, \sigma)$ is a pre-Cartier braided category.

Definition (Pre-Cartier coquasitriangular bialgebra $(H, \mathcal{R}, \chi)$)

An **infinitesimal $\mathcal{R}$-form** of a coquasitriangular bialgebra $(H, \mathcal{R})$ is a linear map $\chi\colon H \otimes H \to \Bbbk$ s.t. $\chi(h_1 \otimes h'_1) h_2 h'_2 = h_1 h'_1 \chi(h_2 \otimes h'_2)$ and

$$\chi(\mathrm{id} \otimes \mu) = \chi_{12} + \mathcal{R}_{12}^{-1} * \chi_{13} * \mathcal{R}_{12}$$
$$\chi(\mu \otimes \mathrm{id}) = \chi_{23} + \mathcal{R}_{23}^{-1} * \chi_{13} * \mathcal{R}_{23},$$

where $*$ denotes the convolution product. We call $(H, \mathcal{R}, \chi)$ pre-Cartier.
If also $\mathcal{R} * \chi = \chi^{\mathrm{op}} * \mathcal{R}$ we call $(H, \mathcal{R}, \chi)$ Cartier.

All statements about pre-Cartier quasitriangular bialgebras admit a dual statement for pre-Cartier coquasitriangular bialgebras.

Quantum 2×2-matrices

Let $q \in \mathbb{C}$ be non-zero. Define $\mathrm{M}_q(2)$ as the free algebra generated by $\alpha, \beta, \gamma, \delta$ modulo the Manin relations

$$\alpha\beta = q\beta\alpha, \qquad \alpha\gamma = q\gamma\alpha, \qquad \beta\delta = q\delta\beta, \qquad \gamma\delta = q\delta\gamma,$$
$$\beta\gamma = \gamma\beta, \qquad \alpha\delta - \delta\alpha = (q - q^{-1})\beta\gamma.$$

Bialgebra structure $\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\varepsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Coquasitriangular structure

$$\mathcal{R} \begin{pmatrix} \alpha\otimes\alpha & \beta\otimes\beta & \alpha\otimes\beta & \beta\otimes\alpha \\ \gamma\otimes\gamma & \delta\otimes\delta & \gamma\otimes\delta & \delta\otimes\gamma \\ \alpha\otimes\gamma & \beta\otimes\delta & \alpha\otimes\delta & \beta\otimes\gamma \\ \gamma\otimes\alpha & \delta\otimes\beta & \gamma\otimes\beta & \delta\otimes\alpha \end{pmatrix} = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 1 & q - q^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition (Ardizzoni-Bottegoni-Sciandra-TW '23)

There is a non-trivial infinitesimal $\mathcal{R}$-form $\chi := \partial_q \otimes \partial_q$ on $(\mathrm{M}_q(2), \mathcal{R})$, making the latter Cartier, i.e. with $\mathcal{R} * \chi = \chi^{\mathrm{op}} * \mathcal{R}$ in addition.

Above, $\partial_q \colon \mathrm{M}_q(2) \to \mathbb{C}$ is defined on homogeneous elements $\xi \in \mathrm{M}_q(2)$ of degree $|\xi|$ by $\partial_q(\xi) := |\xi|\varepsilon(\xi)$. It satisfies the Leibniz rule $\partial_q(\xi\eta) = \partial_q(\xi)\varepsilon(\eta) + \varepsilon(\xi)\partial_q(\eta)$ and is central, i.e. $\partial_q(\xi_1)\xi_2 = \xi_1\partial_q(\xi_2)$.

The quantum groups $\mathrm{GL}_q(2)$ and $\mathrm{SL}_q(2)$

We denote the quantum determinant by $\det_q := \alpha\delta - q\beta\gamma$ and define

$$\mathrm{GL}_q(2) := \mathrm{M}_q(2)[\theta]/(\theta\det_q - 1), \qquad \mathrm{SL}_q(2) := \mathrm{M}_q(2)/(\det_q - 1).$$

It is known that $\mathrm{GL}_q(2), \mathrm{SL}_q(2)$ are coquasitriangular Hopf algebras with bialgebra structure and $\mathcal{R}$ induced from $\mathrm{M}_q(2)$.

Define $\hat{\partial}_q \colon \mathrm{GL}_q(2) \to \mathbb{C}$ by $\hat{\partial}_q|_{\mathrm{M}_q(2)} = \partial_q$ and $\hat{\partial}_q(\theta^n) = -2n$.

Proposition

$(\mathrm{GL}_q(2), \mathcal{R}, \chi)$ *is a Cartier coquasitriangular bialgebra with* $\chi := \hat{\partial}_q \otimes \hat{\partial}_q$.

Note that $\chi = \partial_q \otimes \partial_q$ does **not** descend to $\mathrm{SL}_q(2)$, since e.g.

$$\chi(\alpha \otimes \det_q) = 2\varepsilon(\alpha^2\delta) - 2q\varepsilon(\alpha\beta\gamma) = 2 \neq 0 = \chi(\alpha \otimes 1).$$

Moreover, we show

Proposition

$(\mathrm{SL}_q(2), \mathcal{R})$ *has* **no** *non-trivial infinitesimal braiding.*

FRT construction

A **braided vector space** (V, c) is a vs V with an invertible solution $c \in \mathrm{End}_{\mathbb{k}}(V \otimes V)$ of the braid equation $c_{12}c_{23}c_{12} = c_{23}c_{12}c_{23}$.
Assume V finite-dim. and choose basis $v_1, \ldots, v_N$. Set of indeterminates $\{T_i^j\}_{1 \leq i,j \leq N}$.

$\Rightarrow$ The free algebra $F = \mathbb{k}\{T_i^j\}_{1 \leq i,j \leq N}$ is a bialgebra
with $\Delta(T_i^j) = T_i^k \otimes T_k^j$ (sum over k understood) and $\varepsilon(T_i^j) = \delta_i^j$.

Define $c_{ij}^{k\ell} \in \mathbb{k}$ by $c(v_i \otimes v_j) = c_{ij}^{k\ell} v_k \otimes v_\ell$ and denote by $I(c)$ two-sided ideal in F generated by

$$C_{ij}^{k\ell} := c_{ij}^{mn} T_m^k T_n^\ell - T_i^m T_j^n c_{mn}^{k\ell}$$

and the quotient (bialgebra!) by $A(c) := F/I(c)$. Coaction $v_i \mapsto T_i^j \otimes v_j$.

Theorem (Faddeev-Reshetikhin-Takhtajan '89)

For each finite-dim braided vector space (V, c) there is a unique coquasitriangular structure $\mathcal{R}$ on $A(c)$ s.t. $\mathcal{R}(T_i^k \otimes T_j^\ell) = c_{ji}^{k\ell}$.

Infinitesimally braided vector spaces

We call (V, c, t) **infinitesimally braided vector space** if (V, c) is finite-dim braided vs and $t \in \mathrm{End}_{\Bbbk}(V \otimes V)$ s.t.

$$c_{23} t_{12} c_{23}^{-1} = c_{12}^{-1} t_{23} c_{12} = c_{23}^{-1} t_{12} c_{23} = c_{12} t_{23} c_{12}^{-1}, \tag{1}$$

$$[t_{23}, c_{12} t_{23} c_{12}^{-1}] = [t_{12}, t_{23}] = [c_{23} t_{12} c_{23}^{-1}, t_{12}]. \tag{2}$$

Remark

If c is the tensor flip, then (1) *are trivially satisfied and* (2) *become the* **"infinitesimal pure braid relations"** *[Kohno '87].*

Example

Every braided vs (V, c) is infinitesimally braided via $t = \lambda \cdot \mathrm{id}_{V \otimes V}$, $\lambda \in \Bbbk$.

Example (Braided vs of diagonal type)

Let V be braided vs with $c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i$ for some $q_{ij} \in \Bbbk \setminus \{0\}$.
Then (V, c) is infinitesimally braided with $t(v_i \otimes v_j) = p_{ij} v_i \otimes v_j$ for arbitrary $p_{ij} \in \Bbbk$.
We have also a more general (exhaustive) solution.

Let (V, c, t) be an infinitesimally braided vector space. Denote by $I(t)$ the 2-sided ideal in $A(c)$ generated by

$$D_{ij}^{k\ell} := t_{ij}^{mn} T_m^k T_n^\ell - T_i^m T_j^n t_{mn}^{k\ell}$$

and the quotient algebra by $A(c, t) := A(c)/I(t)$.

Theorem (Ardizzoni-Bottegoni-Sciandra-TW '23, Infinitesimal FRT)

For (V, c, t) there is a unique infinitesimal $\mathcal{R}$-form $\chi\colon A(c,t) \otimes A(c,t) \to \Bbbk$ with $\chi(T_i^k \otimes T_j^\ell) = t_{ij}^{k\ell}$ s.t. $(A(c,t), \mathcal{R}, \chi)$ is pre-Cartier coquasitriangular.

Proposition (Canonical infinitesimal braiding on $(A(c), \mathcal{R})$)

For every braided vs (V, c) the coquasitriangular bialgebra $(A(c), \mathcal{R})$ admits a 1-parameter family of infinitesimal braidings

$$\chi_\lambda(T_{i_1}^{j_1} \dots T_{i_m}^{j_m} \otimes T_{k_1}^{\ell_1} \dots T_{k_n}^{\ell_n}) = \lambda mn \varepsilon(T_{i_1}^{j_1} \dots T_{i_m}^{j_m} T_{k_1}^{\ell_1} \dots T_{k_n}^{\ell_n})$$

such that $(A(c), \mathcal{R}, \chi_\lambda)$ is Cartier. $\mathrm{M}_q(2)$ *example arises this way!*

Proposition

If $(H, \mathcal{R}, \chi)$ is pre-Cartier coquasitriangular and V a left H-comodule, then (V, c, t) is infinitesimally braided vs, where $c(v \otimes v') := \mathcal{R}(v'_{-1} \otimes v_{-1}) v'_0 \otimes v_0$, $t(v \otimes v') := \chi(v_{-1} \otimes v'_{-1}) v_0 \otimes v'_0$.

Some details on the inf. FRT proof

$(H, \mathcal{R})$ coquasitriangular bialgebra. Define left and right actions $\triangleright, \triangleleft \colon H \otimes H \to H$ by

$$a \triangleright b := \mathcal{R}^{-1}(a_1 \otimes b_1) b_2 \mathcal{R}(a_2 \otimes b_3),$$
$$a \triangleleft b := \mathcal{R}^{-1}(a_1 \otimes b_1) a_2 \mathcal{R}(a_3 \otimes b_2).$$

Lemma

The following formulas are satisfied for all $a, b, c \in H$.

$$a \triangleright (bc) = (a_1 \triangleright b_1)\left((a_2 \triangleleft b_2) \triangleright c\right) \quad , \quad (ab) \triangleleft c = (a \triangleleft (b_1 \triangleright c_1))(b_2 \triangleleft c_2),$$
$$\varepsilon(a \triangleleft b) = \varepsilon(a)\varepsilon(b) = \varepsilon(a \triangleright b).$$

$$\chi(a \otimes bc) = \chi(a \otimes b)\varepsilon(c) + \chi(a \triangleleft b \otimes c) \tag{3}$$
$$\chi(ab \otimes c) = \varepsilon(a)\chi(b \otimes c) + \chi(a \otimes b \triangleright c) \tag{4}$$

Proposition

Let C be a coalgebra together with two actions $\triangleright : TC \otimes TC \to TC$ and $\triangleleft : TC \otimes TC \to TC$ such that Lemma above holds true for all $a, b, c \in TC$ and such that $TC \triangleright C^{\otimes n} \subseteq C^{\otimes n}$ and $C^{\otimes n} \triangleleft TC \subseteq C^{\otimes n}$. Then every linear map $\chi_{11} : C \otimes C \to \Bbbk$ induces a linear map $\chi : TC \otimes TC \to \Bbbk$ satisfying (3) and (4).

Theorem (Tannaka-Krein, see e.g. [Ulbrich '90])

$\mathcal{C}$ braided monoidal category, $\omega\colon \mathcal{C} \to \mathrm{Vec}_f$ strong monoidal functor to finite-dim. vector spaces. Then there exists a coquasitriangular bialgebra $(H, \mathcal{R})$ and a strong braided monoidal functor $\Omega\colon \mathcal{C} \to \mathcal{M}^H_f$ s.t.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad\Omega\quad} & \mathcal{M}^H_f \\ & {\scriptstyle\omega}\searrow \quad \swarrow{\scriptstyle F} & \\ & \mathrm{Vec}_f & \end{array}$$

where F is the forgetful functor.

Theorem (Ardizzoni-Bottegoni-Sciandra-TW '23, Infinitesimal Tannaka-Krein)

If $\mathcal{C}$ is a (pre-)Cartier braided monoidal category then the reconstructed bialgebra $(H, \mathcal{R}, \chi)$ is (pre-)Cartier.

Proof.

$\mathrm{Hom}_{\mathbb{k}}(H \otimes H, \mathbb{k}) \cong \mathrm{Nat}(\omega \otimes \omega, \omega \otimes \omega \otimes \mathbb{k})$ and $\chi\colon H \otimes H \to \mathbb{k}$ corresponds to

$$\omega(X) \otimes \omega(Y) \to \omega(X \otimes Y) \xrightarrow{\omega(t_{X,Y})} \omega(X \otimes Y) \to \omega(X) \otimes \omega(Y) \to \omega(X) \otimes \omega(Y) \otimes \mathbb{k}$$

□

Open questions and future research

- **Deformation problem:** We have seen that $\tilde{\mathcal{R}}$ on (trivial) topological bialgebra $\tilde{H} = H[[\hbar]]$ gives infinitesimal braidings χ for $(H, \mathcal{R})$ via

$$\tilde{\mathcal{R}} = \mathcal{R}(1 \otimes 1 + \hbar\chi + \mathcal{O}(\hbar^2)). \tag{5}$$

Given any pre-Cartier quasitr. bialgebra $(H, \mathcal{R}, \chi)$ is there a quasitriangular structure $\tilde{\mathcal{R}}$ on $\tilde{H}$ such that (5) holds, i.e. such that $\tilde{\mathcal{R}}$ "quantizes" χ?

Very preliminary result: for Sweedler's Hopf algebra H we have a quasitriangular structure $\tilde{\mathcal{R}} := \mathcal{R}\exp(\hbar\chi) = \mathcal{R}(1 \otimes 1 + \hbar\chi)$ on $\tilde{H}$.

Remark

By Etingof-Kazhdan this also works for $(U\mathfrak{g}, 1 \otimes 1, \chi = r + r^{\mathrm{op}})$, where $(\mathfrak{g}, r)$ quasitriangular Lie bialgebra (note that bialgebra str. of $\tilde{H}$ is deformed non-trivially!).

- **Noncommutative differential geometry from infinitesimal braidings:** The axioms of $(H, \mathcal{R}, \chi)$ determine a vector field calculus for every (co)module algebra of H. Work in progress: go from symmetric Riemannian geometry (which is well-understood) to braided Riemannian geometry by first solving the "first-order approximation" determined by χ.

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Thank you for your attention!