Morita equivalence for the Erhesmann-Schauenburg Hopf algebroid

Jacopo Zanchettin (SISSA)
Joint work with A. Chirvasitu (UB) and M. Tobolsky (Wroclaw Uni.)

Queen Mary University London (UK)
Hopf Algebroids and Noncommutative Geometry

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1. Preliminaries

2. Hopf algebroids and bibundles

3. A Morita equivalence result

In differential geometry, two Lie groupoids $\mathfrak{G}$ and $\mathfrak{G}'$ are said to be Morita equivalent if there exists a manifold equipped with principal right $\mathfrak{G}$ and left $\mathfrak{G}'$ action. A quite easy example of a Lie groupoid is the gauge (or Atiyah) groupoid associated to a principal bundle.

In differential geometry, two Lie groupoids $\mathfrak{G}$ and $\mathfrak{G}'$ are said to be Morita equivalent if there exists a manifold equipped with principal right $\mathfrak{G}$ and left $\mathfrak{G}'$ action. A quite easy example of a Lie groupoid is the gauge (or Atiyah) groupoid associated to a principal bundle. A Hopf algebroid is a dual object to a groupoid, in the same spirit that Hopf algebras are dual to groups. A Morita theory for *commutative* Hopf algebroids was developed by El Kaoutit and Kowalzig (Doc. Math. 22, 551-609, 2017). In the noncommutative case such a characterization has yet to be done, but the notion of bibundle still makes sense in this context. Very briefly, given two Hopf algebroids $\mathcal{L}$ and $\mathcal{L}'$, a $(\mathcal{L}, \mathcal{L}')$-bibundle is a bicomodule algebra such that the coactions are principal.

Our goal here is to prove the following Lie groupoids result in the Hopf context

Theorem

Let $\mathfrak{G}$ be a Lie groupoid. Then the following are equivalent:

1. *$\mathfrak{G}$ is Morita equivalent to a Lie group.*
2. *$\mathfrak{G}$ is isomorphic to the gauge groupoid associated to a principal bundle.*

Let $\mathbb{K}$ be a field and $\otimes := \otimes_{\mathbb{K}}$. Throughout the slides H denotes a Hopf algebra with coalgebra structure (Δ, ϵ) and antipode S that is always assumed to be invertible.

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A (right) H-comodule algebra A is an algebra equipped with a coaction

$$\rho : A \longrightarrow A \otimes H, \quad a \longmapsto a_{(0)} \otimes a_{(1)}$$

that is an algebra morphism compatible with the coalgebra structure of H.

The space of coaction invariant elements

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$$\mathrm{can} : A \otimes_{A^{coH}} A \longrightarrow A \otimes H, \quad a \otimes_{A^{coH}} \tilde{a} \longmapsto a\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}$$

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is bijective.
We focus on extensions such that A is a faithfully flat A^{coH}-module. We recall that this means that the functor $- \otimes_{A^{coH}} A$ preserves and reflexes exact sequences.

Denote by $\mathcal{M}_{A^{coH}}$ the category of right A^{coH}-modules and $\mathcal{M}_A^H$ the category of right A-module with compatible right H-comodule structure.

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Theorem (Schneider's theorem)

Let H be a Hopf algebra with bijective antipode, then the following are equivalent:

1. *The functor* $- \otimes_{A^{coH}} A : \mathcal{M}_{A^{coH}} \longrightarrow \mathcal{M}_A^H$ *is an equivalence.*
2. *The functor* $A \otimes_{A^{coH}} - : {}_{A^{coH}}\mathcal{M} \longrightarrow {}_A\mathcal{M}^H$ *is an equivalence.*
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The adjoint functor of $- \otimes_{A^{coH}} A$ is given by $V \longmapsto V^{coH}$ with $V \in \mathcal{M}_A^H$. So for faithfully flat Hopf-Galois extensions we have the following isomorphism

$$(M \otimes_{A^{coH}} A)^{coH} \simeq M, \quad V^{coH} \otimes_{A^{coH}} A \simeq V$$

for all $M \in \mathcal{M}_{A^{coH}}$ and $V \in \mathcal{M}_A^H$.

Let now B be an algebra and $B^e := B \otimes B^{op}$. A B^e-**ring** is a triple (U, s, t) where U is an algebra and

$$s : B \longrightarrow U, \quad t : B^{op} \longrightarrow U$$

are algebra morphisms with commuting ranges. In this way it is defined a B-bimodule structure on U via

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A B-**coring** is a triple (V, Δ, ϵ) where V is a B-bimodule and

$$\Delta : V \longrightarrow V \otimes_B V, \quad \epsilon : V \longrightarrow B$$

are B-bimodule morphisms defining a (coassociative) coproduct and counit on V.

Now a (left) B-bialgebroid is a quintuple $(\mathcal{L}, s, t, \Delta, \epsilon)$ where

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- $(\mathcal{L}, \Delta, \epsilon)$ is a B-coring w.r.t. the B-bimodule structure inherited from the B^e-ring structure.
- The coproduct Δ is an algebra morphism if corestricted to the Takeuchi product

$$\mathcal{L} \times_B \mathcal{L} := \{l \otimes_B l' \in \mathcal{L} \otimes_B \mathcal{L} | lt(b) \otimes_B l' = l \otimes_B l's(b), \forall b \in B\}$$

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Moreover the counit ϵ is unital and satisfies an additional requirement we do not use here.

A (left) **Hopf algebroid** $\mathcal{H}$ is a B-bialgebroid $\mathcal{L}$ such that the canonical map

$$\beta : \mathcal{H} \odot_{B^{op}} \mathcal{H} \longrightarrow \mathcal{H} \otimes_B \mathcal{H}, \quad h \odot_{B^{op}} h' \longmapsto h_{(1)} \otimes_B h_{(2)} h'$$

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Remark

In case $B = \mathbb{K}$ one has $\mathcal{H}$ is a bialgebra and this condition is equivalent to the existence of the antipode making $\mathcal{H}$ a Hopf algebra.

A left $\mathcal{H}$-**comodule algebra** is the datum of (P, α) where P is an algebra and $\alpha : B \longrightarrow P$ an algebra morphism, together with a left B-linear map

$$\lambda : P \longrightarrow \mathcal{H} \otimes_B P, \quad p \longmapsto p^{(-1)} \otimes_B p^{(0)}$$

defining a coaction such that its corestriction to $\mathcal{H} \times_B P$ is an algebra morphism.

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Via symmetry one defines right $\mathcal{H}$-comodule algebras. A $(\mathcal{H}, \mathcal{H}')$-bicomodule algebra is a triple (P, α, α') such that (P, α) ia a left $\mathcal{H}$-comodule algebra with right B'-linear coaction λ, (P, α') is a right $\mathcal{H}'$-comodule algebra with left B-linear coaction ρ such that

$$(\mathrm{id}_{\mathcal{H}} \otimes_B \rho) \circ \lambda = (\lambda \otimes_{B'} \mathrm{id}_{\mathcal{H}'}) \circ \rho$$

A principal $(\mathcal{H}, \mathcal{H}')$-**bibundle** is a $(\mathcal{H}, \mathcal{H}')$-bicomodule algebra (P, α, α') such that the extension α and α' are faithfully flat and the canonical maps

$$\mathrm{can}_{\mathcal{H}} : P \otimes_{B'} P \longrightarrow \mathcal{H} \otimes_B P, \quad p \otimes_{B'} p' \longmapsto p^{(-1)} \otimes_B p^{(0)} p'$$
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Remark

For two Lie groupoid $\mathfrak{G}$ and $\mathfrak{G}'$ a principal bibundle is a manifold X endowed with a left $\mathfrak{G}$-action and right $\mathfrak{G}'$-action that commute and moreover the associated canonical maps are bijective. If a bibundle exists $\mathfrak{G}$ and $\mathfrak{G}'$ are said to be **Morita equivalent**.

Attached to any faithfully flat Hopf-Galois extension $A^{coH} \subseteq A$ we have the **Erhesmann-Schauenburg Hopf algebroid** over $B := A^{coH}$. As a vector space it is given by $\mathcal{C}(A, H) := (A \otimes A)^{coH}$, where $A \otimes A$ is a right H-comodule if endowed with

$$\rho^{\otimes} : A \otimes A \longrightarrow A \otimes A \otimes H, \quad a \otimes \tilde{a} \longmapsto a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)}\tilde{a}_{(1)}$$

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Let $\tau := \mathrm{can}^{-1}|_H : H \longrightarrow A \otimes_B A$ be the translation map, i.e. $\tau(h) = 1_A \otimes h$. The following map defines a left $\mathcal{C}(A, H)$-comodule algebra structure on A

$$\lambda : A \longrightarrow \mathcal{C}(A, H) \otimes_B A, \quad a \longmapsto a_{(0)} \otimes \tau(a_{(1)}).$$

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It is compatible with the right H-comodule algebra structure of A and moreover the corresponding canonical map is bijective. Then A is a principal $(\mathcal{C}(A, H), H)$-bibundle.

On the other hand, if we have a left Hopf B-algebroid $\mathcal{L}$ adimitting a $(\mathcal{L}, H)$-bibundle A where H is a Hopf algebra, then $\mathcal{L} \simeq \mathcal{C}(A, H)$.

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$$\mathcal{M}^H(A, V \otimes_B A) \simeq \mathcal{M}_B(\mathcal{C}(A, H), V)$$

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Now let $\mathcal{L}$ be a B-bialgebroid such that A is a $(\mathcal{L}, H)$-bibundle with coaction $\delta : A \longrightarrow \mathcal{L} \otimes_B A$. Because of the above equivalence there exists a unique right B-module map $f : \mathcal{C}(A, H) \longrightarrow \mathcal{L}$ such that $\lambda = (f \otimes_B \mathrm{id}_A) \circ \delta$.

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Now let $\mathcal{L}$ be a B-bialgebroid such that A is a $(\mathcal{L}, H)$-bibundle with coaction $\delta : A \longrightarrow \mathcal{L} \otimes_B A$. Because of the above equivalence there exists a unique right B-module map $f : \mathcal{C}(A, H) \longrightarrow \mathcal{L}$ such that $\lambda = (f \otimes_B \mathrm{id}_A) \circ \delta$. One proves that f is a B-bialgebroid morphism and by the principality of the coactions of both $\mathcal{C}(A, H)$ and $\mathcal{L}$ on A concludes that is invertible since

$$\mathrm{can}_{\mathcal{L}} = (f \otimes_B \mathrm{id}_A) \circ \mathrm{can}_{\mathcal{C}(A,H)}$$

Proposition

Let H be a Hopf algebra with bijective antipode and $\mathcal{L}$ a Hopf algebroid over an algebra B. Then the following are equivalent:

1. *There exists a principal $(\mathcal{L}, H)$-bibundle.*
2. *$\mathcal{L}$ is isomorphic to the Erhesmann-Schauenburg bialgebroid associated to a faithfully flat H-Hopf-Galois extension.*

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Question: can we give the result in terms of category equivalences? More precisely, is true that ${}^{H}\mathcal{M} \simeq {}^{\mathcal{L}}\mathcal{M}$ if and only if $\mathcal{L} \simeq \mathcal{C}(A, H)$?

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Question: can we give the result in terms of category equivalences? More precisely, is true that ${}^{H}\mathcal{M} \simeq {}^{\mathcal{L}}\mathcal{M}$ if and only if $\mathcal{L} \simeq \mathcal{C}(A, H)$? One implication is true, namely if $B \subseteq A$ is a faithfully flat H-Hopf-Galois extension then $A\Box_H - : {}^{H}\mathcal{M} \longrightarrow {}^{\mathcal{L}}\mathcal{M}$ is an (monoidal) equivalence.

Thank you!